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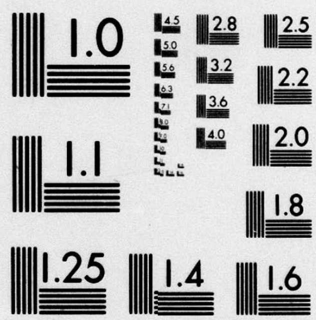

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(6) CONVERGENT APPROXIMATIONS IN PARABOLIC VARIATIONAL INEQUALITIES. I. ONE-PHASE STEFAN PROBLEMS.

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(Received November 27, 1979)

(15) DAAG29-75-C-ØØ24

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CONVERGENT APPROXIMATIONS IN PARABOLIC VARIATIONAL INEQUALITIES. I: ONE-PHASE STEFAN PROBLEMS

Joseph W. Jerome

Technical Summary Report #2032
January 1980

ABSTRACT

The change of variable

$$u(x,t) = \int_0^t \theta(x,\tau)\,d\tau \tag{1}$$

for the temperature θ in the one-phase Stefan problem leads to the evolution inequality

$$(u_t - \Delta u - f)(z - u) \geq 0, \quad \text{for all} \quad z \geq 0 \quad , \tag{2}$$

where $u \geq 0$ is required. This inequality is to hold over a space-time domain $D = \Omega \times (0,T_0)$ with a Dirichlet boundary condition imposed on $\partial\Omega \times (0,T_0)$ and a zero initial condition. In this paper we examine semi-discretizations of (2) in time and in space and we derive the respective convergence rates. The following explicit results are obtained:

$$\|U^M - u\|_{L^2(D)} \leq C\Delta t \quad , \tag{3i}$$

where U^M is the $H^1(\Omega)$-valued, piecewise linear in time, interpolant obtained from the horizontal line, fully implicit Euler scheme applied to (2) with $\Delta t = T_0/M$;

$$\|U_h - u\|_{L^2(D)} \leq Ch^2 \quad , \tag{3ii}$$

where U_h is the continuous time, finite element approximation obtained by applying an integrated version of (2) to a translate of the finite dimensional trial space of C° piecewise linear elements. The approximation scheme used to define U_h appears to be new.

AMS(MOS) Subject Classification - 65M20, 65N30, 76V05

Key Words - Parabolic variational inequalities, one-phase Stefan problem, horizontal line method, finite element method.

Work Unit Number 2 - Physical Mathematics

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

SIGNIFICANCE AND EXPLANATION

Many physical phenomena are modelled by inequalities rather than equations. In this report we examine a dynamic, or parabolic inequality, which characterizes the change of phase of a substance, e.g., ice melting, under certain assumptions. We obtain rates of convergence for certain approximation schemes which separately discretize time and space. The rates, as well as one of the schemes, appear to constitute new results in a subject which is still rather undeveloped. Subsequent investigations are also contemplated for inequalities related to different physical models.

CONVERGENT APPROXIMATIONS IN PARABOLIC VARIATIONAL INEQUALITIES. I: ONE-PHASE STEFAN PROBLEMS

Joseph W. Jerome

§1. Introduction.

Parabolic variational inequalities arise in various modelling problems such as phase transition via heat conduction, optimal stopping time problems, dynamic obstacle problems and porous medium filtration problems. It is the purpose of this series of papers to define and analyze convergent discretizations for the aforementioned problems beginning in this first paper with the one-phase Stefan problem, formally equivalent to a dynamic obstacle problem. Although it might be preferred to treat all of these problems in a single theory, the special features of each make the unified convergence analysis awkward at best. In order to avoid subsequent repetition, we shall restrict ourselves in this paper to the analysis of semidiscretizations in space and time.

The one-phase Stefan problem has the formulation over a fixed space-time domain D,

$$\text{(1.1)}\qquad \begin{aligned} &\text{(i)} && u \geq 0 \ , \\ &\text{(ii)} && u_t - \Delta u - f \geq 0 \ , \\ &\text{(iii)} && (u_t - \Delta u - f)u = 0 \ , \end{aligned}$$

subject to appropriate initial and boundary conditions (cf. Duvaut [4], Bensoussan and Lions [1], Friedman and Kinderlehrer [6]). The specification and development of this formulation will be given later in the introduction together with a statement of the physical problem.

An exhaustive numerical analysis of (1.1) and other variational inequalities has been given by Glowinski, Lions and Trémolières [8], though rates of convergence, and certainly optimal rates of convergence, do not appear to have been derived in the literature. For (1.1), we give a horizontal line analysis and a continuous time, finite-element analysis. We find it convenient, however, to work with the equivalent

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formulation (cf. Theorem 2.4),

$$
\begin{aligned}
&\text{(i)} && u \geq 0 \quad , \\
(1.2) \qquad &\text{(ii)} && (Tu)_t + (u - r) - T\Delta r - Tf \geq 0 \quad , \\
&\text{(iii)} && [(Tu)_t + (u - r) - T\Delta r - Tf]u = 0 \quad ,
\end{aligned}
$$

on D, where r specifies the boundary values of u and $T = (-\Delta)^{-1}$ is a smoothing operator. The semidiscretizations are defined directly in terms of (1.2). In terms of the corresponding stationary or elliptic formulation associated with (1.1), this amounts to minimizing

$$
(1.3) \qquad E(u) = \frac{1}{2}(u,u)_{L^2(\Omega)} - (r,u)_{L^2(\Omega)} - (\Delta r + f,u)_{H^{-1}(\Omega)}
$$

over a convex set in $L^2(\Omega)$ rather than the usual quadratic energy form over a convex set in $H^1(\Omega)$. Now in terms of the horizontal line, discrete time approximations, the use of (1.2) merely provides an efficient way of estimating the rate of convergence, $O(\Delta t)$, in $L^2(D)$ (cf. Theorem 3.2). In this case, the approximations based on (1.1) and (1.2) are identical. However, if (1.2) is used to define finite element, continuous time approximations taken from the translate, $r - \mathring{M}_h$, of the C° piecewise linear trial space $\mathring{M}_h$, with homogeneous boundary values, it is <u>possible</u> that the L^2 projections associated with (1.2) are not identical to the H^1 projections defined by (1.1). For equations, the projections coincide. Thus, for elliptic equations, replacement of the energy by (1.3) leaves the finite element approximation invariant, so long as the discrete H^{-1} norm is properly defined. This carries over to parabolic equations (cf. [10]) but remains an open question for inequalities.

In fact, the previous discussion has been highly simplified, since it proceeds as if Ω were a polyhedral domain, compatible with the vanishing of elements of $\mathring{M}_h$ on $\partial\Omega$. Actually, we shall assume that Ω is smoothly bounded to take advantage of the known regularity of u in this case; thus, there is an accompanying boundary layer effect, well-known to be associated with $h^{3/2}$ convergence order in L^2. We

shall avoid the boundary layer by using the L^2 formulation of (1.2) to define finite element approximations indirectly (cf. (4.9)) in terms of non-negative definite, self-adjoint approximations T_h of T, which map into M_h rather than $\mathring{M}_h$, i.e., the trial functions are unrestricted on $\partial\Omega$. The triangulation, thus, includes non-simplicial elements near $\partial\Omega$. Of course, this removes any possibility that the L^2 projection coincides with the H^1 projection, but does permit the derivation of $O(h^2)$ convergence in $L^2(D)$, (cf. Theorem 4.3), provided the operators T_h are pointwise non-negative, which we explicitly assume (cf. (4.8i)). This always holds for T(cf. Lemma 2.1) and, since T_h is defined as the composition of T and the H^1 projection E_h, always holds for T_h in one space dimension, since E_h coincides with the interpolation operator. It is of course possible that the pointwise non-negativity hypothesis on the T_h masks certain boundary layer effects in several Euclidean dimensions. In analyzing the finite element approximations defined by the integration method just outlined, the reader should not confuse this method with direct methods of least squares type, which effectively double the order.

It is of some interest to compare the approach to the one-phase Stefan problem using variational inequalities with that using the classical formulation

$$(1.4)\qquad \begin{array}{ll} \text{(i)} & \theta_t = \Delta\theta, \quad \theta > 0 \ , \\ \text{(ii)} & \theta = 0, \ \text{on } S \ , \\ \text{(iii)} & A \cos(\nu, 1_t) = [\frac{\partial\theta}{\partial\nu}], \ \text{across } S \ , \end{array}$$

for the temperature θ of one of the two phases in a two-phase system such as water-ice. The change of phase results from heat conduction through the phase Ω^t of positive temperature, which is separated from the phase $\Omega\backslash\bar{\Omega}^t$ of fixed temperature 0 by the moving boundary; S is the time profile of this moving phase transition boundary, A is the latent heat of fusion, ν is the outward normal to $\{(x,t) : \theta(x,t) = 0\}$ and $[\frac{\partial\theta}{\partial\nu}]$ represents the discontinuity of $\frac{\partial\theta}{\partial\nu}$ across S directed by ν. The fundamental numerical work on (1.4) has been carried out by

Meyer (cf. [12]) who devised uniformly convergent free boundary approximation schemes. An alternative approach, using the variational inequality formulation (1.1), obtained by setting

$$u(x,t) = \int_0^t \theta(x,\tau)d\tau \quad ,$$

and

$$f(x) = \begin{cases} \theta(x,0) & , \quad x \in \Omega_0 \quad , \\ -A & , \quad x \notin \Omega_0 \quad , \end{cases}$$

was given by the writer in [9]. It appears that the two approaches are equivalent with respect to asymptotic rates of convergence for the computation of $S : \{x = x(t),t\}$ in one space dimension. Indeed, for horizontal line semidiscretizations, errors in computing x are of the order of the square root of errors in computing u; the latter is computed with accuracy Δt, giving a net accuracy of $\sqrt{\Delta t}$. If the less regular variable θ is used, the error in computing θ is of order $\sqrt{\Delta t}$, yet accuracy in x is proportional to accuracy in θ; the net result is again accuracy of order $\sqrt{\Delta t}$. This heuristic analysis has been confirmed in analytical studies of the one-phase Stefan problem by L. Caffarelli.

In the remainder of this section we shall discuss the precise formulation of the variational inequality to be used in the sequel. It can be shown (cf. [9]) that (1.1) is equivalent to a pointwise cone inequality (cf. Abstract) and thence to the integrated version of this same inequality. More precisely, let homogeneous initial datum and a boundary datum function r be specified satisfying

(1.5)
$$\text{(i)} \quad r \in X = W^{1,\infty}(0,T_0;L^\infty(\Omega)) \cap L^\infty(0,T_0;W^{2,\infty}(\Omega)) \quad ,$$
$$\text{(ii)} \quad r(\cdot,0) = 0 \quad , \quad r > 0 \ \text{in} \ D \ .$$

Define a convex set C by

(1.6) $$C = \{w \in X : \ w \le r \ \text{in} \ D \ , \ w(\cdot,t) \in H_0^1(\Omega) \ , \ 0 < t < T_0\} \ .$$

Finally, let f be given satisfying

(1.7) $$f \in L^{\infty}(D) \quad .$$

Then a dynamical obstacle formulation of a parabolic variational inequality with solution v may be used to characterize u. It may be written, incorporating initial and boundary conditions, as

$$\text{(1.8)}\quad \begin{aligned} &\text{(i)} \quad u = r - v \ , \\ &\text{(ii)} \quad v \in \mathcal{C} \ , \ v(\cdot,0) = 0 \ , \\ &\text{(iii)} \quad \int_\Omega (v_t - \Delta v - f_0)(w - v) \geq 0, \ \text{ for all } w \in \mathcal{C}, \ \text{a.e. in } (0,T_0) \ . \end{aligned}$$

Here,

(1.9) $$f_0 = r_t - \Delta r + f \quad .$$

Remark 1.1. The formulation (1.8) is convenient for piecewise linear finite elements. It is immediately seen that (1.8) is equivalent to

$$\text{(1.10)}\quad \begin{aligned} &\text{(i)} \quad u \in r - \mathcal{C} \ , \ u(\cdot,0) = 0 \ , \\ &\text{(ii)} \quad \int_\Omega (u_t - \Delta u - f)(z - u) \geq 0, \ \text{ for all } z \in r - \mathcal{C}, \ \text{a.e. in } (0,T_0) \end{aligned}$$

which in turn is equivalent to (1.1), augmented by the initial and boundary conditions, and to the non-integrated form of (1.10).

Remark 1.2. Standard methods (cf. Caffarelli and Friedman [3]) show that (1.10) has a unique solution provided f is C^{α}, $0 < \alpha < 1$, in each of connected subdomains Ω_0 and Ω_1 with common smooth boundary. In the case of the one-phase Stefan problem, Ω_0 and Ω_1 are the regions occupied initially by, say, water and ice. In [3] it is also shown that $\theta = u_t$ is continuous on D.

Remark 1.3. Throughout the paper we shall assume that (1.10) has a unique solution $u \in r - \mathcal{C}$. In addition, we shall require some further regularity properties of u and Ω. The hypothesis on u is described in (1.11) below. The hypotheses on Ω are basically the existence of classical (Schauder) regularity theory and weak maximum principles (cf. Lemmas 2.1, 2.2) as well as L^2 regularity theory (cf. (2.2)). We introduce (1.11) now. Although u_{tt} exists only as a measure in D, it is to be

expected on the basis of explicit modulus of continuity estimates in [3], coupled with the well-known representation (cf. John [11]) for fundamental solution kernels corresponding to $-\Delta$, that

$$(Tu)_{tt} \in L^2(D) \quad . \tag{1.11}$$

Here, once again, $T = (-\Delta)^{-1}$ when $-\Delta$ is viewed as an operator from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. The idea of replacing the hypothesis $u_{tt} \in L^2(D)$ by the weaker hypothesis (1.11) was introduced by the author and M. Rose in [10], where it is in fact proved that $[TH(u)]_{tt} \in L^2(D)$ for the discontinuous enthalpy $H(\cdot)$.

§2. Positivity and Equivalent Formulation.

Denote by T the inverse of $-\Delta$ introduced in section one. Thus T is an isomorphism of $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$. Also,

$$(\ell_1, \ell_2)_{H^{-1}(\Omega)} = \langle T\ell_1, \ell_2 \rangle \tag{2.1}$$

defines an inner product and norm on $H^{-1}(\Omega)$, with the latter equivalent to the standard norm. Here, $\langle \cdot,\cdot \rangle$ denotes the duality pairing on $H_0^1(\Omega) \times H^{-1}(\Omega)$. T is well-known to be a positive-definite, self-adjoint operator in $L^2(\Omega)$ when restricted to this space. One of the basic hypotheses on Ω is contained in our assumption that T is an isomorphism of $H^{m-2}(\Omega)$ onto $H^m(\Omega) \cap H_0^1(\Omega)$ for $m \geq 1$:

$$\|Tg\|_{H^m(\Omega)} \leq C\|g\|_{H^{m-2}(\Omega)} \quad , \quad m \geq 1 \ . \tag{2.2}$$

As a first step in obtaining the formulation (1.2) we state the following familiar type of maximum principle.

<u>Lemma 2.1</u>. T has the positivity property in the sense of Korovkin:

$$g \in L^2(\Omega) \quad , \quad g \geq 0 \Rightarrow Tg \geq 0 \quad , \tag{2.3}$$

where the inequalities hold a.e. in Ω.

Proof: Follows directly from [1, Théorème 5.1, p. 83] or, alternatively, from the classical maximum principle [7, p. 35] applied to the dense subset $C^\infty(\bar{\Omega})$ of $L^2(\Omega)$.

<u>Lemma 2.2</u>. Suppose u satisfies (1.10). Then u satisfies the initial condition,

$$u(\cdot,0) = 0 \quad , \tag{2.4}$$

and for all $0 < t < T_0$, the boundary condition,

$$u(\cdot,t) = r(\cdot,t), \text{ on } \partial\Omega \quad , \tag{2.5}$$

and the integrated inequality

$$\begin{aligned} &\text{(i)} \quad u \geq 0, \text{ in } \Omega \ , \\ &\text{(ii)} \quad (Tu)_t + u - T\Delta r - r - Tf \geq 0, \text{ in } \Omega \ , \\ &\text{(iii)} \quad \int_\Omega [(Tu)_t + u - T\Delta r - r - Tf]u = 0 \ . \end{aligned} \tag{2.6}$$

Proof: (2.4), (2.5) and (2.6i) are immediate and (2.6ii) follows from Lemma 2.1 by applying T to

(2.7) $$u_t - \Delta u - f \geq 0 \quad .$$

Note that here we use the equivalent characterization (1.1). Now note that by (2.6i,ii) it suffices to prove

(2.8) $$\int_\Omega [(Tu)_t + u - T\Delta r - r - Tf]u \leq 0 \quad .$$

Thus, if

(2.9) $$\Omega_+^t = \{x \in \Omega : u(x,t) > 0\} \quad ,$$

then Ω_+^t is open. Suppose t is fixed and Ω_* is any ball in Ω_+^t satisfying $\bar{\Omega}_* \subset \Omega_+^t$. In particular, $u(x,t) \geq c > 0$ for $x \in \Omega_*$. For almost all $t \in (0,T_0)$, $u(\cdot,t) \in C^1(\bar{\Omega})$ so that, by the Schauder theory [7], $Tu(\cdot,t) \in C^{2+\alpha}(\bar{\Omega})$, some $0 < \alpha < 1$. In particular, if $\psi \in C_0(\Omega_*)$, $\psi \geq 0$, there is an $\varepsilon > 0$ such that

$$\varepsilon\psi T(u(\cdot,t)) < c, \quad \text{on } \Omega_* \quad .$$

Thus, the choice

$$z = u - \varepsilon\psi Tu$$

leads to $z \in r - \mathcal{C}$ and

$$\int_\Omega (u_t - \Delta u - f)\psi Tu \leq 0 \quad .$$

Since ψ is arbitrary, we may set $\psi = \psi_n$ where $\psi_n \to 1$ in $L^2(\Omega)$. This leads to

$$\int_\Omega (u_t - \Delta u - f)Tu \leq 0$$

and then immediately to (2.8) for almost all $t \in (0,T_0)$, and hence for all t by standard continuity. ∎

<u>Lemma 2.3</u>. Suppose u satisfies (1.10). Then u satisfies (2.4), (2.5) and (1.2) for all $0 < t < T_0$.

Proof: It is enough to prove that

(2.10) $$\{T[u_t - \Delta u - f]\}u \leq 0 \quad \text{in } \Omega_+^t \quad , \quad 0 < t < T_0 \quad ,$$

where Ω_+^t is given by (2.9). It follows from (2.6) that

$$\int_\Omega \{T[u_t - \Delta u - f]\}(z - u) \geq 0 \tag{2.11}$$

for all

$$z = u + \omega \quad , \quad \text{supp}\ \omega \subset \Omega \quad , \quad 0 \leq z \in L^2(\Omega) \quad .$$

If Ω_* is any compact subset of Ω_+^t and $z = u - \varepsilon u \chi_{\Omega_*}$, $0 < \varepsilon < 1$, we have from (2.11),

$$0 \geq \varepsilon \int_\Omega \{T[u_t - \Delta u - f]\} u \chi_{\Omega_*}$$

so that (2.10) holds. ■

<u>Theorem 2.4</u>. The solution of (1.2), (1.10i) is unique. In particular, it is given by the solution of (1.10).

Proof: If u_1 and u_2 are solutions of (1.2), (1.10i), then

$$\{T[\frac{\partial u_1}{\partial t} - \Delta u_1 - f]\}(u_1 - u_2) \leq 0 \quad ,$$

$$\{T[\frac{\partial u_2}{\partial t} - \Delta u_2 - f]\}(u_1 - u_2) \geq 0 \quad .$$

Subtraction and integration over Ω gives

$$(\frac{\partial(u_1 - u_2)}{\partial t}, u_1 - u_2)_{H^{-1}(\Omega)} + (u_1 - u_2, u_1 - u_2)_{L^2(\Omega)} \leq 0 \quad .$$

Integration in t gives

$$\frac{1}{2} \| u_1(\cdot,t) - u_2(\cdot,t) \|^2_{H^{-1}(\Omega)} + \int_0^t \| u_1 - u_2 \|^2_{L^2(\Omega)} ds \leq 0 \quad . \tag{2.12}$$

for $0 < t < T$. Uniqueness is immediate from (2.12) and the remaining statement follows from Lemma 2.3.

<u>Remark 2.1</u>. Lemmas 2.2 and 2.3 form the basis of a circle of equivalences for the solution u of (1.10) as in section one. We note for future reference that the solution of (1.10) may be characterized by

$$(2.13)\qquad \begin{aligned} &\text{(i)}\quad u \in r - C \ , \quad u(\cdot,0) = 0 \ , \\ &\text{(ii)}\quad \int_{\Omega} [(Tu)_t + (u - r) - T\Delta r - Tf](z - u) \geq 0 \ , \end{aligned}$$

for all $0 < t < T_0$ and all $z \in L^2(\Omega)$, $z \geq 0$.

§3. Horizonal Line Analysis.

Definition 3.1. Let M be a positive integer, let $\Delta t = T_0/M$ and set $t_m = m\Delta t$, $0 \le m \le M$. With $u_0 = 0$ the given initial datum, let $\{u_m\}_{m=0}^{M}$ be the recursively generated sequence of solutions of the elliptic variational inequalities

$$
(3.1)\qquad
\begin{aligned}
&(i)\quad u_m \in r_m - C_m\ , \\
&(ii)\quad \int_\Omega \left(\frac{u_m - u_{m-1}}{\Delta t} - \Delta u_m - f_m \right)(\varphi - u_m) \ge 0, \quad \text{for all } \varphi \in C_m\ ,
\end{aligned}
$$

obtained formally from (1.7) by the identifications $u(\cdot,t_m) \sim u_m$ and $\frac{\partial u}{\partial t}(t_m) \sim \frac{u_m - u_{m-1}}{\Delta t}$. Here

$$
(3.2)\qquad
\begin{aligned}
&(i)\quad r_m = r(\cdot,t_m)\ , \quad f_m = f(\cdot,t_m)\ , \\
&(ii)\quad C_m = \{\psi \in W^{2,\infty}(\Omega) : \psi \le r_m\}\ .
\end{aligned}
$$

Remark 3.1. A unique solution $u_m \in W^{2,\infty}(\Omega)$ of (3.1) is known to exist, provided $f(\cdot,t)$ is piecewise C^α in subdomains of Ω independent of t and separated by a smooth boundary, as described in the introduction. Although the result of Frehse [5] assumes a global C^α property for $f(\cdot,t)$, modifications (cf. [9]) show that the piecewise result is true. This is, of course, the case of physical interest. Appropriate modifications of the result of Brezis and Kenderlehrer [2] also yield the result. The technical difficulty involves the global boundedness of the second derivatives.

Lemma 3.1. The relation

$$
(3.3)\qquad \int_\Omega \left[\frac{Tu_m - Tu_{m-1}}{\Delta t} + u_m - T\Delta r(\cdot,t_m) - r_m - f_m \right] u_m = 0
$$

holds for all $1 \le m \le M$.

Proof: This proceeds by a straightforward repetition of the proof of Lemma 2.2. ∎

Theorem 3.2. There exists a constant C, independent of Δt and given by (3.13) below, such that

$$(3.4)\qquad \max_{0\le m\le M} \|u(\cdot,t_m) - u_m\|^2_{H^{-1}(\Omega)} + \sum_{m=0}^{M} \|u(\cdot,t_m) - u_m\|^2_{L^2(\Omega)} \Delta t \le C(\Delta t)^2 .$$

In particular, if $u_{s,\Delta t}$ is the step function

$$(3.5)\qquad u_{s,\Delta t}(x,t) = u_m(x) \quad , \quad x \in \Omega \quad , \quad m\Delta t \le t < (m+1)\Delta t \quad ,$$

for $0 \le m \le M - 1$, then the estimate

$$(3.6)\qquad \|u - u_{s,\Delta t}\|_{L^2(D)} \le C(\Delta t)$$

holds for some constant C.

Proof: Setting $t = t_m$ in (2.6iii) gives, since $u_m \ge 0$,

$$(3.7)\qquad \int_\Omega [\frac{\partial Tu}{\partial t}(\cdot,t_m) + u(\cdot,t_m) - T\Delta r(\cdot,t_m) - r_m - f_m](u(\cdot,t_m) - u_m) \le 0 .$$

Similarly, we obtain from (3.3) the reverse inequality,

$$(3.8)\qquad \int_\Omega [\frac{Tu_m - Tu_{m-1}}{\Delta t} + u_m - T\Delta r(\cdot,t_m) - r_m - f_m](u(\cdot,t_m) - u_m) \ge 0 .$$

For simplicity, we set

$$y_m = u(\cdot,t_m) - u_m .$$

Then, subtraction of (3.8) from (3.7) yields, after an appropriate addition and subtraction,

$$\frac{1}{\Delta t}(y_m,y_m)_{H^{-1}(\Omega)} - \frac{1}{\Delta t}(y_{m-1},y_m)_{H^{-1}(\Omega)} + \|y_m\|^2_{L^2(\Omega)}$$

$$\le \int_\Omega [\frac{Tu(\cdot,t_m) - Tu(\cdot,t_{m-1})}{\Delta t} - \frac{\partial Tu}{\partial t}(t_m)]y_m$$

$$(3.9)\qquad = \frac{1}{\Delta t}\int_\Omega \{\int_{t_{m-1}}^{t_m} [\frac{\partial Tu}{\partial t}(\tau) - \frac{\partial Tu}{\partial t}(t_m)]d\tau\}y_m$$

$$= \int_\Omega \{\frac{-1}{\Delta t}\int_{t_{m-1}}^{t_m}\int_\tau^{t_m} \frac{\partial^2 Tu}{\partial t^2}(\cdot,\sigma)d\sigma d\tau\}y_m$$

$$= (y_m,z_m)_{L^2(\Omega)} ,$$

where z_m is defined by this equation. Multiplying through by Δt, making use of the inequality

$$(3.10) \qquad |(w,z)| \leq \frac{1}{2}\|w\|^2 + \frac{1}{2}\|z\|^2$$

and summing on $m = 1, \ldots, k$ yields

$$(3.11) \qquad \frac{1}{2}\|y_k\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\sum_{m=1}^{k}\|y_m\|^2_{L^2(\Omega)}\Delta t \leq \frac{1}{2}\sum_{m=1}^{k}\|z_m\|^2_{L^2(\Omega)}\Delta t .$$

It remains to estimate the right hand side of (3.11). We have, by the Cauchy-Schwarz inequality and the limiting integral form of the triangle inequality,

$$(3.12) \qquad \|z_m\|^2 \leq \Big[\frac{1}{\Delta t}\int_{t_{m-1}}^{t_m}\int_{\tau}^{t_m}\Big\|\frac{\partial^2 Tu}{\partial t^2}(\cdot,\sigma)\Big\|^2_{L^2(\Omega)}d\sigma d\tau\Big]^2$$

$$\leq \Delta t\Big\|\frac{\partial^2 Tu}{\partial t^2}\Big\|^2_{L^2(t_{m-1},t_m;L^2(\Omega))}$$

so that (3.4) follows with

$$(3.13) \qquad C = \Big\|\frac{\partial^2 Tu}{\partial t^2}\Big\|^2_{L^2(D)} .$$

Finally, (3.6) follows via the triangle inequality from (3.4) and the estimate

$$(3.14) \qquad \sum_{m=0}^{M-1}\int_{t_m}^{t_{m+1}}\int_\Omega |u(\cdot,t) - u(\cdot,t_m)|^2 = \sum_{m=0}^{M-1}\int_{t_m}^{t_{m+1}}\int_\Omega \Big|\int_{t_m}^{t}\frac{\partial u}{\partial t}(\tau)d\tau\Big|^2$$

$$\leq \Delta t\sum_{m=0}^{M-1}\int_{t_m}^{t_{m+1}}\|u_t\|^2_{L^2(t_m,t_{m+1};L^2(\Omega))}$$

$$= \Delta t^2\|u_t\|^2_{L^2(D)} .$$

The theorem is now proved. ■

§4. Continuous Time Finite Element Approximations.

For $h > 0$, let $\mathcal{T}_h$ be a triangulation of the given domain Ω. Thus,

$$\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}_h} \tau \tag{4.1}$$

where τ is a typical (closed) element in the simplicial decomposition $\mathcal{T}_h$; in particular, we permit nonsimplical elements near the boundary. Let M_h denote the linear space of continuous piecewise linear trial functions determined by $\mathcal{T}_h$:

$$M_h = \{\chi \in C(\bar{\Omega}) : \chi|_\tau \text{ is linear } \forall \tau \in \mathcal{T}_h\} . \tag{4.2}$$

Let E_h be the Ritz-Galerkin $H^1(\Omega)$ projection defined by

$$(E_h\varphi, \chi)_{H^1(\Omega)} = (\varphi, \chi)_{H^1(\Omega)}, \quad \text{for all } \chi \in M_h , \tag{4.3}$$

for each fixed $\varphi \in H^1(\Omega)$; here we use

$$(\varphi, \psi)_{H^1(\Omega)} = (\nabla\varphi, \nabla\psi)_{L^2(\Omega)} + \frac{1}{|\Omega|} \int_\Omega \varphi \int_\Omega \psi , \tag{4.4}$$

which defines a norm equivalent to the standard $H^1(\Omega)$ norm in the usual way (cf. [13]). One sees easily that E_h preserves (integral) mean values over Ω and that E_h is an orthogonal projection in $H^1(\Omega)$. Finally, define the finite rank approximation T_h of T by

$$T_h = E_h \circ T \quad , \quad T_h : H^{-1}(\Omega) \to M_h . \tag{4.5}$$

Remark 4.1. The mapping T_h, defined by (4.5), is not the natural finite rank projection associated with T; this would be obtained by defining E_h with respect to

$$(\varphi, \psi)_{H_0^1(\Omega)} = (\nabla\varphi, \nabla\psi)_{L^2(\Omega)} , \tag{4.6}$$

leading to a self-adjoint, positive definite operator on $\mathring{M}_h$. It is therefore somewhat surprising that T_h is self-adjoint and non-negative definite on $L^2(\Omega)$.

Lemma 4.1. T_h is self-adjoint on $L^2(\Omega)$ and satisfies

$$(4.7) \qquad (T_h g,g)_{L^2(\Omega)} \geq 0, \quad \text{for all} \quad g \in L^2(\Omega) \quad .$$

Proof: Since T_h is bounded, the self-adjointness will follow from the symmetry of T_h. This, together with (4.7), follow from the following identities. Since T is a Riesz mapping,

$$(T_h g_1, g_2)_{L^2(\Omega)} = (\nabla E_h T g_1, \nabla T g_2)_{L^2(\Omega)}$$

so that, by (4.4), the mean value property of E_h, and its role as an orthogonal projection in $H^1(\Omega)$,

$$\begin{aligned}(T_h g_1, g_2)_{L^2(\Omega)} &= (E_h T g_1, T g_2)_{H^1(\Omega)} - \frac{1}{|\Omega|} \int_\Omega E_h T g_1 \int_\Omega T g_2 \\ &= (E_h T g_1, E_h T g_2)_{H^1(\Omega)} - \frac{1}{\Omega} \int_\Omega E_h T g_1 \int_\Omega E_h T g_2 \\ &= (\nabla E_h T g_1, \nabla E_h T g_2)_{L^2(\Omega)} \quad .\end{aligned}$$

The latter quantity is non-negative if $g_1 = g_2 = g$ and the interchange of g_1 and g_2 yields the symmetry of T_h. ∎

<u>Remark 4.2</u>. Our basic finite element hypotheses are the following. For $g \in L^2(\Omega)$

$$(4.8) \qquad \begin{aligned} &\text{(i)} \quad g \geq 0 \Rightarrow T_h g \geq 0 \quad \text{(Korovkin Positivity)} \quad ; \\ &\text{(ii)} \quad \|(T - T_h)g\|_{L^2(\Omega)} \leq Ch^2 \|g\|_{L^2(\Omega)} \end{aligned}$$

holds for C independent of h. Note that (4.8ii) is a routine consequence of (2.2) for $m = 2$ since, if $w = Tg$, then $w_h = T_h g$ satisfies $w_h = E_h w$. The order h^2 approximation properties in L^2 are well-known in this case (cf. [14]).

We are now ready to define the continuous time finite element approximation U_h. The approximation is based upon (2.13) and the intrinsic approximation properties of the operators T_h.

<u>Definition 4.1</u>. The finite element, continuous time approximation $U_h : [0,T_0] \rightarrow r - C \cap M_h$ is defined to be the unique solution of the initial value problem

(4.9i) $$\int_\Omega [(T_h U_h)_t + (U_h - r) - T_h r - T_h f](\chi - U_h) \geq 0$$

for all $\chi \in r - C \cap M_h$, $0 < t < T_0$,

(4.9ii) $$U_h(0) = 0 \quad .$$

Remark 4.3. Of course, it requires an argument to prove that (4.9) possesses a unique solution. Uniqueness is established in a manner similar to that of the proof of Theorem 2.4; in particular, (cf. (2.12)) it can be proved that

(4.10) $$\frac{1}{2} (T_h[U_{1,h} - U_{2,h}], U_{1,h} - U_{2,h})_{L^2(\Omega)}$$

$$+ \int_0^t \|U_{1,h} - U_{2,h}\|^2_{L^2(\Omega)} ds \leq 0$$

for any two solutions $U_{1,h}$ and $U_{2,h}$ of (4.9), where (4.10) holds for each $0 < t < T_0$. Uniqueness follows from (4.10) and (4.7). An existence proof can be constructed using the method of horizontal lines, wherein the time derivative in (4.9i) is replaced by a backward divided difference, quite similar to the technique in section three. There result stationary problems which are the gradient formulations of quadratic minimization problems over closed convex subsets of $L^2(\Omega)$. The step function (in time) sequence indexed by $M = T/\Delta t$ can be shown to be bounded in $L^2(\Omega)$ (actually, the finite dimensional affine subspace $r - M_h$) and convergence of a subsequence to a solution of (4.9) may be established by standard methods.

Lemma 4.2. Under the hypothesis (4.8i), the relation (4.9) holds for all $\chi \in r - C$ and in particular for $\chi = u$.

Proof: Since T_h is self-adjoint, (4.9i) is equivalent to

(4.11) $$\int_\Omega \{[(U_h)_t - f](T_h\chi - T_h U_h) + \nabla U_h \cdot \nabla(T_h\chi - T_h U_h)\} \geq 0$$

for all $\chi \in r - C \cap M_h$. We rewrite (4.11) as

(4.12) $$\int_\Omega \{[(U_h)_t - f](\zeta - T_h U_h) + \nabla U_h \cdot \nabla(\zeta - T_h U_h)\} \geq 0 \quad ,$$

for all $\zeta \in T_h r - T_h(C \cap M_h) = T_h M_h^+$. The lemma will therefore follow, if we can

prove that

(4.13) $$T_h(r - C) = T_hM_h^+ ,$$

from the self-adjointness of T_h. Here we have written

(4.14) $$M_h^+ = \{\chi \in M_h : \chi \geq 0\} .$$

Since T_h is assumed pointwise non-negative by (4.8i), (4.13) will follow if, for $v \in L^2(\Omega)$,

(4.15) $$\{\psi \in M_h, \psi \perp T_hM_h\} \Rightarrow \{\psi \perp T_hv\} .$$

Here, the orthogonality in (4.15) is understood to be L^2 orthogonality. To verify (4.15), let $v \in L^2(\Omega)$, let $\psi \in M_h$ and suppose

$$\psi \perp T_hM_h .$$

By the proof of Lemma 4.1 we have

$$0 = (T_h\psi,\psi)_{L^2(\Omega)} = \|\nabla T_h\psi\|^2_{L^2(\Omega)}$$

so that $T_h\psi$ is constant on Ω. Thus, using the aforementioned proof once again we have,

$$0 = (\nabla T_hv,\nabla T_h\psi)_{L^2(\Omega)} = (T_hv,\psi)_{L^2(\Omega)}$$

which verifies (4.15). The proof is completed.

We are now ready for the major result of this section.

<u>Theorem 4.3</u>. The finite element approximations defined by (4.9) are convergent to the solution u of (1.10) with order h^2 in $L^2(D)$, i.e.,

(4.16) $$\|u - U_h\|_{L^2(D)} \leq Ch^2$$

for some constant C.

Proof: Setting $z = U_h$ in (2.13ii) and $\chi = u$ in (4.9i) (cf. Lemma 4.2) we have

$$\text{(4.17)}\qquad \begin{aligned} &\text{(i)} \quad \int_\Omega [(Tu)_t + (u - r) - T\Delta r - Tf](u - U_h) \le 0 \\ &\text{(ii)} \quad \int_\Omega [(T_h U_h)_t + (U_h - r) - T_h \Delta r - T_h f](u - U_h) \ge 0 \ . \end{aligned}$$

Subtraction of (4.17ii) from (4.17i) gives, after some rearrangement,

$$\text{(4.18)}\qquad \begin{aligned} &\int_\Omega \{T_h(u - U_h)_t \cdot (u - U_h)\} + \int_\Omega (u - U_h)^2 \\ &\quad \le \int_\Omega (u - U_h)(T_h - T)u_t + \int_\Omega (u - U_h)(T - T_h)(f + \Delta r) \ . \end{aligned}$$

Writing

$$\text{(4.19)}\qquad \int_\Omega \{T_h(u - U_h)_t \cdot (u - U_h)\} = \frac{1}{2}\frac{d}{dt} \int_\Omega T_h(u - U_h) \cdot (u - U_h) \ ,$$

integrating in t from 0 to T_0 and using the non-negative definiteness of T_h yield

$$\text{(4.20)}\qquad \begin{aligned} \|u - U_h\|^2_{L^2(D)} &\le \frac{1}{2} \|u - U_h\|^2_{L^2(D)} \\ &+ \frac{1}{2} \|(T - T_h)(u_t - f - \Delta r)\|^2_{L^2(D)} \end{aligned}$$

where we have applied the standard inequality (3.10) to the right hand side of (4.18). The estimate (4.16) now follows immediately from (4.20) and (4.8ii) applied to $g = u_t - f - \Delta r$. The proof is now concluded.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 2032	2. GOVT ACCESSION NO. AD-A083 814	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Convergent Approximations in Parabolic Variational Inequalities. I: One-Phase Stefan Problems		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Joseph W. Jerome		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 2 - Physical Mathematics
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE January 1980
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Parabolic variational inequalities, one-phase Stefan problem, horizontal line method, finite element method.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The change of variable

(1) $$u(x,t) = \int_0^t \theta(x,\tau)d\tau$$

for the temperature θ in the one-phase Stefan problem leads to the evolution inequality

(2) $$(u_t - \Delta u - f)(z - u) \geq 0, \quad \text{for all } z \geq 0 ,$$

where $u \geq 0$ is required. This inequality is to hold over a space-time

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ABSTRACT (Continued)

domain $D = \Omega \times (0,T_0)$ with a Dirichlet boundary condition imposed on $\partial\Omega \times (0,T_0)$ and a zero initial condition. In this paper we examine semi-discretizations of (2) in time and in space and we derive the respective convergence rates. The following explicit results are obtained:

(3i) $$\|U^M - u\|_{L^2(D)} \le C\Delta t \quad ,$$

where U^M is the $H^1(\Omega)$-valued, piecewise linear in time, interpolant obtained from the horizontal line, fully implicit Euler scheme applied to (2) with $\Delta t = T_0/M$;

(3ii) $$\|U_h - u\|_{L^2(D)} \le Ch^2 \quad ,$$

where U_h is the continuous time, finite element approximation obtained by applying an integrated version of (2) to a translate of the finite dimensional trial space of C° piecewise linear elements. The approximation scheme used to define U_h appears to be new.